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Spectral Scale of Self-adjoint Operators and Trace Inequalities

DÉNES PETZ

Mathematical Institute of the Hungarian Academy of Sciences, Budapest, Hungary

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Localization of the eigenvalues of matrices goes back to the very beginning of matrix theory. Inequalities including some convexity properties have had applications in various fields as well as in mathematics. To be more concrete, we take a simple example from [10]. According to von Neumann a state of a quantum system can be described with a density matrix ρ. The entropy $-\operatorname{tr} \rho \log \rho$ is a concave function of the density ρ (as it should be for physical reasons). Von Neumann pointed out that this property is due to the concavity of the function $t \mapsto -t \cdot \log t$, that is, the functional $\rho \mapsto \operatorname{tr} f(\rho)$ is concave for any concave function f (see also [1, p. 25]).

In this paper we extend some well-known notions and classic inequalities of matrix theory and convex analysis to an operator algebra context. Motivated by the generalized singular values of Fack ([7]) we introduce the spectral scale of a self-adjoint operator in a finite von Neumann algebra and we use it to obtain trace inequalities in C^*-algebras. We prove the above result of von Neumann in a C^*-algebra and we find a generalization of the Klein convexity inequality ([5, 13]).

Let $\mathscr{A} \subset \mathscr{B}(\mathscr{H})$ be a finite von Neumann algebra with a fixed faithful normal trace τ. We assume that $\tau(1) = 1$. If a is a self-adjoint operator affiliated to $\mathscr{A}$ then we define its spectral scale as

$$\lambda_t(a) = \min\{s \colon \tau(p^a(s, \infty)) \leqslant t\} \qquad (t \in [0, 1])$$

where $\int t\, dp^a(t)$ is the spectral resolution of a. It is clear that $\lambda_{\cdot}(a)$ is continuous from the right and non-increasing.

If ω is an eigenvalue of a then $\lambda_t(a) = \omega$ for $s \leqslant t < s + d$, where $d = \tau(p^a(\omega))$ and $s = \tau(p^a(\omega, \infty))$. Conversely, any constant piece of the graph of $\lambda_{\cdot}(a)$ indicates an element of the point spectrum. More generally, $\lambda_{\cdot}(a)\colon [0, 1] \to \mathbb{R}$ is measure preserving if $[0, 1]$ is endowed with the Lebesgue measure and the Borel measure $H \mapsto \tau(p^a(H))$ is regarded on $\mathbb{R}$. This fact implies immediately the following

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PROPOSITION 1. *If $a \in \mathscr{A}^{sa}$ and f is a bounded Borel function of $\mathbb{R}$ then*

$$\tau(f(a)) = \int_0^1 f(\lambda_t(a))\, dt.$$

The next theorem is a minimax characterization of the spectral scale (cf. [7, 1.3, Proposition]).

THEOREM 1. *For $a \in \mathscr{A}^{sa}$ and $0 \leqslant t \leqslant 1$ we have*

$$\lambda_t(a) = \inf \sup\{\langle a\xi, \xi\rangle : \xi \in \mathscr{H},\ \|\xi\| = 1,\ e\xi = \xi\}$$

where inf *is taken over all the projections e such that $\tau(e^\perp) \leqslant t$.*

Proof. We denote by $\omega_t(a)$ the right hand side for a while. Choosing the appropriate spectral projection of a we have $\omega_t(a) \leqslant \lambda_t(a)$. Assume that $\omega_t(a) < \lambda_t(a)$. Then there exists $e \in \mathscr{A}^p$ such that $\tau(e^\perp) \leqslant t$ and

$$\sup\{\langle a\xi, \xi\rangle : e\xi = \xi,\ \|\xi\| = 1\} = \alpha < \beta < \lambda_t(a).$$

Since $\tau(e^\perp) \leqslant t < \tau(p^a(\beta, \infty))$ we have $e \wedge p^a(\beta, \infty) \neq 0$ and we can find $\xi \in \mathscr{H}$ such that $\|\xi\| = 1$, $p^a(\beta, \infty)\, \xi = \xi$ and $e\xi = \xi$. Then $\langle a\xi, \xi\rangle \geqslant \beta$, which is a contradiction.

COROLLARY. *If $a, b \in \mathscr{A}^{sa}$ and $a \leqslant b$ then*

$$\lambda_t(a) \leqslant \lambda_t(b)$$

for every $t \in [0, 1]$.

THEOREM 2. *If $f\colon \mathbb{R} \to \mathbb{R}$ is a monotone function then $a \mapsto \tau(f(a))$ is monotone on $\mathscr{A}^{sa}$.*

Proof. The previous corollary and Proposition 1 give the claim.

Now we make some remarks. For $a \in \mathscr{A}^+$ one can see that $\lambda_t(a) = \mu_t(a)$, where $\mu_t(a)$ is the tth singular value defined in [7]. So

$$\lambda_t(a) = \inf\{\|ae\| : e \in \mathscr{A}^p,\ \tau(e^\perp) \leqslant t\}.$$

On the other hand, $\lambda_t(b + \omega \cdot 1) = \lambda_t(b) + \omega$ for every $b \in \mathscr{A}^{sa}$ and $\omega \in \mathbb{R}$. This property enables us to deduce estimations for $\lambda_t(b)$ from formulas on $\mu_t(b)$.

PROPOSITION 2. *Let $a, b \in \mathscr{A}^{sa}$ and $v \in \mathscr{A}$. Then*

(i) $\lambda_{t+s}(a + b) \leqslant \lambda_t(a) + \lambda_s(b)$ $(t, s \geqslant 0,\ t + s \leqslant 1)$.

(ii) $|\lambda_t(a)-\lambda_t(b)|\leqslant \|a-b\|$ $(0\leqslant t\leqslant 1)$.

(iii) *If* $0\leqslant t\leqslant 1$, $\lambda_t(a)\geqslant 0$ *then* $\lambda_t(v^*av)\leqslant \|v\|^2\,\lambda_t(a)$.

Proof. We follow the argument in [7]. Assume that $\lambda_t(a)<\alpha$ and $\lambda_s(b)<\beta$. Then we have projections $e, f\in\mathscr{A}$ such that $\tau(e^\perp)\leqslant t$, $\tau(f^\perp)\leqslant s$ and $\langle eae\xi,\xi\rangle\leqslant\alpha\,\|\xi\|^2$, $\langle fbf\xi,\xi\rangle\leqslant\beta\,\|\xi\|^2$ $(\xi\in\mathscr{H})$. We take $p=e\wedge f$. So $\tau(p^\perp)\leqslant\tau(e^\perp)+\tau(f^\perp)\leqslant s+t$ and

$$\langle p(a+b)\,p\xi,\xi\rangle\leqslant(\alpha+\beta)\,\|\xi\|^2.$$

This gives (i).

Case (ii) follows from the trivial estimation

$$\langle eae\xi,\xi\rangle\leqslant\langle ebe\xi,\xi\rangle+\|a-b\|\cdot\|\xi\|^2.$$

To prove (iii) we take $\alpha>\lambda_t(a)$ and a projection $e\in\mathscr{A}$ such that $\tau(e^\perp)\leqslant t$ and $\langle eae\xi,\xi\rangle\leqslant\alpha\,\|\xi\|^2$ $(\xi\in\mathscr{H})$. Let $p=1-\operatorname{supp} e^\perp v=\operatorname{Ker} e^\perp v$. So $p^\perp=\operatorname{supp} e^\perp v\sim\operatorname{supp} v^*e^\perp\leqslant e^\perp$ and we infer $\tau(p^\perp)\leqslant\tau(e^\perp)\leqslant t$. Since $e^\perp vp=0$ we have

$$\langle pv^*avp\xi,\xi\rangle=\langle avp\xi, vp\xi\rangle\leqslant\alpha\,\|vp\xi\|^2\leqslant\alpha\,\|v\|^2\,\|\xi\|^2$$

and $\lambda_t(v^*av)\leqslant\|v\|^2\cdot\alpha$.

Our next proposition generalizes an estimate of Powers ([11]). We recall that $\|a\|_1=\tau((a^*a)^{1/2})$ and for self-adjoint $b\in\mathscr{A}$ we have simply $\|b\|_1=\tau(b_1)+\tau(b_2)$ if $b=b_1-b_2$ is the Jordan decomposition of b.

PROPOSITION 3. *Let* $a, b\in\mathscr{A}^{sa}$. *Then*

$$\|a-b\|_1\geqslant\int_0^1|\lambda_t(a)-\lambda_t(b)|\,dt.$$

Proof. Let $x-y$ be the Jordan decomposition of $a-b$ and take $c=b+x=a+y$. Then $\|a-b\|_1=2\tau(c)-\tau(a+b)$ and we have

$$\|a-b\|_1=\int_0^1 2\lambda_t(c)-\lambda_t(a)-\lambda_t(b)\,dt.$$

Since $2\lambda_t(c)-\lambda_t(a)-\lambda_t(b)\geqslant|\lambda_t(a)-\lambda_t(b)|$ we obtain the proposition.

For $a\in\mathscr{A}^{sa}$ and $0\leqslant t\leqslant 1$ we define

$$\sigma_t(a)=\int_0^t\lambda_s(a)\,ds.$$

It is straightforward that the function $t\mapsto\sigma_t(a)$ is continuous and concave. It follows from Proposition 2 that for fixed t, $a\mapsto\sigma_t(a)$ is norm continuous.

LEMMA 1. *Let $a \in \mathscr{A}^+$ and $0 < t < 1$. There is a contraction $c \in \mathscr{A}^+$ such that $[a, c] = 0$ and $\tau(ac) = \int_0^t \lambda_s(a)\, ds$. This c may be choosen of the form $c = p_1 + \delta p_2$, where $\tau(c) = t$ and p_1, p_2 are orthogonal projections commuting with α. In addition, if $\mathscr{A}$ is continuous or $\lambda_\cdot(a)$ is not constant in a neighbourhood of t then one can choose $\delta = 0$.*

Proof. If $\lambda_\cdot(a)$ is not constant in t then $\lambda_\cdot(a)^{-1}(\lambda_t(a), \infty) = [0, t)$ and

$$\int_{\lambda_t(a)}^{\infty} s\, d\tau(p^a)(s) = \int_0^t \lambda_s(a)\, ds.$$

So $c = p^a((\lambda_t(a), \infty))$ will do.

If $\lambda_\cdot(a)$ is constant in t then we take the least number t' such that $\lambda_\cdot(a)$ is constant on $[t', t]$. By the argument above

$$\int_0^{t'} \lambda_t(a)\, dt = \tau(ap_1)$$

if $p_1 = p^a((\lambda_{t'}(a), \infty))$. Now $\lambda_{t'}(a)$ is in the point spectrum of a. If $\mathscr{A}$ is continuous then there exists a projection $p_2 \leqslant p^a(\{\lambda_{t'}(a)\})$ such that $\tau(p_2) = (t - t')\, \lambda_t(a) < \tau(p^a(\{\lambda_{t'}(a)\}))$. Then $p_1 + p_2$ is a projection. Otherwise, we can take $c = p_1 + \delta \cdot p^a(\{\lambda_t(a)\})$, where $\delta = (t - t')/\tau(p^a(\{\lambda_t(a)\}))$.

THEOREM 3. *For $a \in \mathscr{A}^{sa}$ and $0 \leqslant t \leqslant 1$ we have*

$$\sigma_t(a) = \sup\{\tau(ac) \colon c \in \mathscr{A}_1^+,\ \tau(c) = t\}.$$

Proof. We may assume that $a \geqslant 0$. For a projection $p \in \mathscr{A}$ with $\tau(p) = t$ we have

$$\tau(ap) = \int_0^1 \lambda_s(pap)\, ds = \int_0^t \lambda_s(pap)\, ds \leqslant \int_0^t \lambda_s(a)\, ds = \sigma_t(a).$$

If $c = \sum_{i=1}^n \delta_i p_i$ is a convex combination of projections then using the concavity of $\sigma_\cdot(a)$ we infer

$$\tau(ac) = \sum_{i=1}^n \delta_i \tau(ap_i) \leqslant \sum_{i=1}^n \delta_i \sigma_{\tau(p_i)}(a) \leqslant \sigma_{\tau(c)}(a).$$

Finally, an approximation argument gives that $\tau(ac) \leqslant \sigma_{\tau(c)}(a)$ for arbitrary $c \in \mathscr{A}_1^+$.

The rest of the theorem is contained in the previous lemma. In addition, Lemma 1 provides that in the case of a continuous algebra it is sufficient to take the sup over the projections.

COROLLARY. *For fixed* $0 \leqslant t \leqslant 1$ *the functional* $a \mapsto \sigma_t(a)$ *is convex on* $\mathscr{A}^{sa}$.

Now we need an old result of Hardy, Littlewood and Pólya ([9]), and we present a simple proof (see also [7, 4.1] and [8, II. Lemma 3.4]).

LEMMA 2. *Let* $g_1, g_2 \colon [0, 1] \to \mathbb{R}$ *be non-increasing functions and let* $f \colon \mathbb{R} \to \mathbb{R}$ *be an increasing convex continuously differentiable function. Assume that*

$$\varphi_1(s) = \int_0^s g_1(t)\, dt \leqslant \int_0^s g_2(t)\, dt = \varphi_2(s) \qquad (0 \leqslant s \leqslant 1).$$

Then

$$\int_0^s f \circ g_1(t)\, dt \leqslant \int_0^s f \circ g_2(t)\, dt \qquad (0 \leqslant s \leqslant 1).$$

Proof. For arbitrary $s_1, s_2 \in [0, 1]$ we have $f(s_2) - f(s_1) = (s_2 - s_1) f'(x)$ with an x between s_1 and s_2. Since f' is non-decreasing we infer

$$f(s_2) - f(s_1) \geqslant (s_2 - s_1) f'(s_1).$$

Applying this and using Stieltjes integrals we find

$$\begin{aligned}\int_0^s f \circ g_2(t) - f \circ g_1(t)\, dt &\geqslant \int_0^s [g_2(t) - g_1(t)]\, f' \circ g_1(t)\, dt \\ &= \int_0^s f' \circ g_1(t)\, d(\varphi_2 - \varphi_1)(t) \\ &= f' \circ g_1(s)[\varphi_2(s) - \varphi_1(s)] \\ &\quad - \int_0^s (\varphi_2 - \varphi_1)(t)\, df' \circ g_1(t).\end{aligned}$$

Here $f' \circ g_1(s)[\varphi_2(s) - \varphi_1(s)] \geqslant 0$ and $\int_0^s (\varphi_2 - \varphi_1)(t)\, df' \circ g_1(t) \leqslant 0$ since $\varphi_2 \geqslant \varphi_1$ and $f' \circ g_1$ is non-increasing. The proof is complete.

THEOREM 4. *Let f be a continuous convex function on* $[\alpha, \beta]$. *Then the functional* $a \mapsto \tau(f(a))$ *is convex on the set* $\{a \in \mathscr{A}^{sa} \colon \operatorname{Sp} a \subset [\alpha, \beta]\}$.

Proof. We have to show that

$$\tau(f(\omega_1 a + \omega_2 b)) \leqslant \omega_1 \tau(f(a)) + \omega_2 \tau(f(b))$$

if $\omega_1, \omega_2 > 0$, $\omega_1 + \omega_2 = 1$ and $\mathrm{Sp}\, a$, $\mathrm{Sp}\, b \subset [\alpha, \beta]$. Since both sides of this inequality are continuous in f (with respect to the uniform convergence on $[\alpha, \beta]$) we may assume that f is continuously differentiable. If η is large enough then $f_0(t) = f(t) + \eta \cdot t$ is increasing and fulfills the requirements of the lemma. Let

$$g_1(t) = \lambda_t(\omega_1 a + \omega_2 b) \qquad \text{and} \qquad g_2(t) = \omega_1 \lambda_t(a) + \omega_2 \lambda_t(b).$$

By the convexity of $\sigma_t(\cdot)$ we obtain

$$\int_0^s g_1(t)\, dt \leqslant \int_0^s g_2(t)\, dt \qquad (0 \leqslant s \leqslant 1)$$

and can apply Lemma 2.

$$\begin{aligned} \tau(f_0(\omega_1 a + \omega_2 b)) &= \int_0^1 f_0(\lambda_t(\omega_1 a + \omega_2 b))\, dt \\ &\leqslant \int_0^1 f_0(\omega_1 \lambda_t(a) + \omega_2 \lambda_t(b))\, dt \\ &\leqslant \int_0^1 \omega_1 f_0(\lambda_t(a)) + \omega_2 f_0(\lambda_t(b))\, dt \\ &= \omega_1 \tau(f_0(a)) + \omega_2 \tau(f_0(b)). \end{aligned}$$

What we have inferred is equivalent to the claim.

Now we are going to extend Theorems 2 and 4 to a C^*-algebra setting by means of GNS construction.

LEMMA 3. *Let $\mathscr{A}$ be a C^*-algebra with a tracial state φ. Then there exists a *-homomorphism $\pi: \mathscr{A} \to \mathscr{B}$ into a von Neumann algebra $\mathscr{B}$ possessing a faithful normal trace τ such that for every $a \in \mathscr{A}^{sa}$ and for every continuous function f: $\mathrm{Sp}\, a \to \mathbb{R}$ we have*

$$\varphi(f(a)) = \tau(f(\pi(a))).$$

Moreover, $\mathscr{B}$ can be assumed to be continuous.

Proof. Let $\{\pi_\varphi, \mathscr{H}_\varphi, \xi_\varphi\}$ be the cyclic representation obtained by GNS construction. Then ξ_φ gives rise to a faithful normal trace τ on $\pi_\varphi(\mathscr{A})''$ (see [15, p. 343]). Clearly,

$$\varphi(f(a)) = \tau(\pi_\varphi(f(a))) = \tau(f(\pi_\varphi(a)))$$

so we can choose $\mathscr{B} = \pi_\varphi(\mathscr{A})''$. If we want to have a continuous algebra $\mathscr{B}$

we embed $\pi_\varphi(\mathscr{A})''$ into a continuous one. For example, $\mathscr{B} = \pi_\varphi(\mathscr{A})'' \bar{\otimes} R_1$, where R_1 is the hyperfinite II_1 factor and one can take the product trace.

THEOREM 5. *Let $\mathscr{A}$ be a C^*-algebra with a tracial positive functional φ. If $f: [\alpha, \beta] \to \mathbb{R}$ is a continuous monotone (convex) function then the functional*

$$a \mapsto \varphi(f(a))$$

is monotone (convex) on the set $\{a \in \mathscr{A}^{sa}: \operatorname{Sp} a \subset [\alpha, \beta]\}$.

Proof. If $a \leqslant b$ then applying Theorem 2 and Lemma 3 we have $\varphi(f(a)) = \tau(f(\pi(a))) \leqslant \tau(f(\pi(b))) = \varphi(f(b))$ for any continuous increasing function f. The case of convex f is similar, one needs Theorem 4.

COROLLARY. *Let $\mathscr{A}$ and $\mathscr{B}$ be C^*-algebras and $\phi: \mathscr{A} \to \mathscr{B}$ a positive linear map such that $\phi(ab) = \phi(ba)$ for every $a, b \in \mathscr{A}$. If $f: [\alpha, \beta] \to \mathbb{R}$ is a continuous function and $D = \{a \in \mathscr{A}^{sa}: \operatorname{Sp} a \subset [\alpha, \beta]\}$ then*

(i) *For $a, b \in D$ we have $\phi(f(a)) \leqslant \phi(f(b))$ provided that $a \leqslant b$ and f is non-decreasing.*

(ii) *For $a, b \in D$ we have $\phi(f((a+b)/2)) \leqslant (1/2)[\phi(f(a)) + \phi(f(b))]$ provided that f is convex.*

We note that the center-valued trace of a finite von Neumann algebra is an important example for so called tracial ϕ in the Corollary (cf. [4]).

Now we prove a generalization of the Klein inequality.

THEOREM 6. *Let $\mathscr{A}$ be a C^*-algebra with a tracial state φ. Assume that $f_i: [\alpha, \beta] \to \mathbb{R}$ and $g_i: [\gamma, \delta] \to \mathbb{R}$ are continuous functions $(i = 1, 2)$ such that*

$$f_1(x)\, g_1(y) \leqslant f_2(x) + g_2(y)$$

for every $\alpha \leqslant x \leqslant \beta$ and $\gamma \leqslant y \leqslant \delta$. Then for $a, b \in \mathscr{A}^{sa}$ we have

$$\varphi(f_1(a)\, g_1(b)) \leqslant \varphi(f_2(a)) + \varphi(g_2(b))$$

if $\operatorname{Sp} a \subset [\alpha, \beta]$ and $\operatorname{Sp} b \subset [\gamma, \delta]$.

Proof. In the light of Lemma 3 we may assume that $\mathscr{A}$ is a continuous von Neumann algebra and φ is a faithful normal trace. $f_1(\cdot) \cdot g_1(\cdot)$ is continuous in the strong operator topology on bounded sets ([6, p. 922]) and so are $\varphi(f_1(\cdot)\, g_1(\cdot))$, $\varphi(f_2(\cdot))$ and $\varphi(g_2(\cdot))$. Hence it is sufficient to prove the inequality on a strongly dense set. Let $a = \sum_{i=1}^n \nu_i p_i$ and $b = \sum_{i=1}^n \mu_i q_i$, where (p_i) and (q_i) are orthogonal families of projections and $\varphi(p_i) = \varphi(q_i) = 1/n$. We rearrange the indices such that $f_1(\nu_1) \geqslant f_1(\nu_2) \geqslant \cdots \geqslant f_1(\nu_n)$

and $g_1(\mu_1) \geqslant g_1(\mu_2) \geqslant \cdots \geqslant g_1(\mu_n)$. Now use the Abel transformation and apply Theorem 3 to estimate

$$\begin{aligned}\varphi(f_1(a)\, g_1(b)) &= f_1(v_n)\, \varphi(g_1(b)) \\ &\quad + \sum_{i=1}^{n-1} (f_1(v_i) - f_1(v_{i+1}))\, \varphi\left(\sum_{j=1}^{i} p_j g_1(b)\right) \\ &\leqslant n^{-1} f_1(v_n) \sum_{i=1}^{n} g_1(\mu_i) \\ &\quad + n^{-1} \sum_{i=1}^{n-1} (f_1(v_i) - f_1(v_{i+1})) \sum_{j=1}^{i} g_1(\mu_j) \\ &= n^{-1} \sum_{i=1}^{n} f_1(v_i)\, g_1(\mu_i) \leqslant n^{-1} \sum_{i=1}^{n} (f_2(v_i) + g_2(\mu_i)) \\ &= \varphi(f_2(a)) + \varphi(g_2(b)).\end{aligned}$$

The C^*-version of the Klein convexity inequality is the following.

COROLLARY. *Let $\mathscr{A}$ be a C^*-algebra with a tracial state φ. If $f\colon (\alpha, \beta) \to \mathbb{R}$ is a continuously differentiable convex function then for every $a, b \in \mathscr{A}^{sa}$ with* $\operatorname{Sp} a$, $\operatorname{Sp} b \subset (\alpha, \beta)$ *we have*

$$\varphi(f(a) - f(b) - (a-b)\, f'(b)) \geqslant 0.$$

Proof. Since f is convex it satisfies the inequality

$$x \cdot f'(y) \leqslant f(x) + y \cdot f'(y) - f(y)$$

for every $x, y \in (\alpha, \beta)$. So Theorem 6 does the rest.

Finally, we make some bibliographical comments. The definition of the spectral scale is influenced by [7], and its properties are similar to those of the "singular values" (see the remark after Theorem 2). Most of the inequalities appearing here are treated in the nice notes [5] in the finite dimensional case, and one also can find physical applications in [5]. Concerning matrices and compact operators, we refer the reader [2] and [8]. It is interesting to note that Berezin ([3]) proved the inequality in Theorem 2 under the superfluous assumption of the convexity of f.

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